Newton's Law of Cooling

A special type of differential equation is Newton's Law of Cooling. In this law, instead of the rate of cooling being proportional to the temperature of the substance, it is proportional to the *difference* between the temperature of the substance and the temperature of its surroundings.

The law can be modelled by the equation: $\frac{dT}{dt} = r(T - T_a)$,
where T_a is the temperature of the surroundings.
The general solution to this equation is: $T - T_a = Ae^{rt}$.

Example: The chemistry teacher pours herself a cup of coffee with water boiling at 100 ^{0}C. She measures the temperature in the lab to be 17 ^{0}C. After 3 minutes, her coffee is 80 ^{0}C. How long will it take to cool to 50 ^{0}C?

$$\frac{dT}{dt} = r(T - T_a) \quad \rightarrow \quad T - T_a = Ae^{rt}$$

- The temperature in the lab is 17 ^{0}C, so $T_a = 17$. So the general equation is now $T - 17 = Ae^{rt}$.

- Since the initial ($t = 0$) temperature of the coffee is 100 ^{0}C:
  $$100 - 17 = Ae^{r(0)}$$
  $$83 = A \cdot 1$$
  So $A = 83$ and the general equation is now $T - 17 = 83e^{rt}$.

- Since the temperature of the coffee is 80 ^{0}C after 3 minutes:
  $$80 - 17 = 83e^{r(3)} \quad \rightarrow \quad 63 = 83e^{3r} \quad \rightarrow \quad {}^{63}\!/_{83} = e^{3r}$$
  $$\ln\left({}^{63}\!/_{83}\right) = \ln(e^{3r}) = 3r \cdot \ln(e) = 3r$$
  $$r = \tfrac{1}{3} \cdot \ln\left({}^{63}\!/_{83}\right) = -0.0919$$
  So the general equation is now $T - 17 = 83e^{-0.0919t} \quad \rightarrow \quad T = 83e^{-0.0919t} + 17$

- To find out how long it takes for the coffee to cool to 50 ^{0}C, substitute $T = 50$ and solve for t: $50 = 83e^{-0.0919t} + 17 \quad \rightarrow \quad e^{-0.0919t} = {}^{33}\!/_{83} \quad \rightarrow \quad t \approx 10$ mins.

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